演習問題

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[問 1]　角運動量演算子について、以下の問に答えよ。

(1)　$[\hat{x}, \hat{p}_x] = i\hbar$, $[\hat{y}, \hat{p}_y] = i\hbar$, $[\hat{z}, \hat{p}_z] = i\hbar$ を用いて、交換関係

$$[\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z, \qquad [\hat{L}_y, \hat{L}_z] = i\hbar\hat{L}_x, \qquad [\hat{L}_z, \hat{L}_x] = i\hbar\hat{L}_y,$$

を示せ。$[\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z$ についてのみ示せばよい。

(2)　交換関係

$$[\hat{L}_z, \hat{x}] = i\hbar\hat{y}, \qquad [\hat{L}_z, \hat{y}] = -i\hbar\hat{x}, \qquad [\hat{L}_z, \hat{z}] = 0,$$
$$[\hat{L}_z, \hat{p}_x] = i\hbar\hat{p}_y, \qquad [\hat{L}_z, \hat{p}_y] = -i\hbar\hat{p}_x, \qquad [\hat{L}_z, \hat{p}_z] = 0,$$

を示せ。

(3)　交換関係

$$[\hat{L}_z, \hat{r}^2] = 0, \qquad [\hat{L}_z, \hat{p}^2] = 0,$$

を示せ。ただし $\hat{r}^2 := \hat{x}^2 + \hat{y}^2 + \hat{z}^2$, $\hat{p}^2 := \hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2$ である。

[解 1]

(1)

$$[\hat{L}_x, \hat{L}_y] = [\hat{y}\hat{p}_z - \hat{z}\hat{p}_y, \hat{z}\hat{p}_x - \hat{x}\hat{p}_z] = [\hat{y}\hat{p}_z, \hat{z}\hat{p}_x] - [\hat{y}\hat{p}_z, \hat{x}\hat{p}_z] - [\hat{z}\hat{p}_y, \hat{z}\hat{p}_x] + [\hat{z}\hat{p}_y, \hat{x}\hat{p}_z],$$

であるが、右辺のそれぞれの項は

$$\begin{aligned}
[\hat{y}\hat{p}_z, \hat{z}\hat{p}_x] &= \hat{z}[\hat{y}\hat{p}_z, \hat{p}_x] + [\hat{y}\hat{p}_z, \hat{z}]\hat{p}_x = 0 + [\hat{y}\hat{p}_z, \hat{z}]\hat{p}_x = (\hat{y}[\hat{p}_z, \hat{z}] + [\hat{y}, \hat{z}]\hat{p}_z)\,\hat{p}_x = -i\hbar\hat{y}\hat{p}_x, \\
[\hat{y}\hat{p}_z, \hat{x}\hat{p}_z] &= \hat{x}[\hat{y}\hat{p}_z, \hat{p}_z] + [\hat{y}\hat{p}_z, \hat{x}]\hat{p}_z = 0 + 0 = 0, \\
[\hat{z}\hat{p}_y, \hat{z}\hat{p}_x] &= \hat{z}[\hat{z}\hat{p}_y, \hat{p}_x] + [\hat{z}\hat{p}_y, \hat{z}]\hat{p}_x = 0 + 0 = 0, \\
[\hat{z}\hat{p}_y, \hat{x}\hat{p}_z] &= \hat{x}[\hat{z}\hat{p}_y, \hat{p}_z] + [\hat{z}\hat{p}_y, \hat{x}]\hat{p}_z = \hat{x}[\hat{z}\hat{p}_y, \hat{p}_z] + 0 = \hat{x}\left(\hat{z}[\hat{p}_y, \hat{p}_z] + [\hat{z}, \hat{p}_z]\hat{p}_y\right) = i\hbar\hat{x}\hat{p}_y,
\end{aligned}$$

であるから、

$$[\hat{L}_x, \hat{L}_y] = i\hbar\left(\hat{x}\hat{p}_y - \hat{y}\hat{p}_x\right) = i\hbar\hat{L}_z,$$

となる。他も同様である。

(2)

$$
\begin{aligned}
[\hat{L}_z, \hat{x}] &= [\hat{x}\hat{p}_y - \hat{y}\hat{p}_x, \hat{x}] = [\hat{x}\hat{p}_y, \hat{x}] - [\hat{y}\hat{p}_x, \hat{x}] = 0 - \hat{y}[\hat{p}_x, \hat{x}] = i\hbar\hat{y}, \\
[\hat{L}_z, \hat{y}] &= [\hat{x}\hat{p}_y - \hat{y}\hat{p}_x, \hat{y}] = [\hat{x}\hat{p}_y, \hat{y}] - [\hat{y}\hat{p}_x, \hat{y}] = \hat{x}[\hat{p}_y, \hat{y}] - 0 = -i\hbar\hat{x}, \\
[\hat{L}_z, \hat{z}] &= [\hat{x}\hat{p}_y - \hat{y}\hat{p}_x, \hat{z}] = [\hat{x}\hat{p}_y, \hat{z}] - [\hat{y}\hat{p}_x, \hat{z}] = 0 - 0 = 0, \\
[\hat{L}_z, \hat{p}_x] &= [\hat{x}\hat{p}_y - \hat{y}\hat{p}_x, \hat{p}_x] = [\hat{x}\hat{p}_y, \hat{p}_x] - [\hat{y}\hat{p}_x, \hat{p}_x] = [\hat{x}, \hat{p}_x]\hat{p}_y - 0 = i\hbar\hat{p}_y, \\
[\hat{L}_z, \hat{p}_y] &= [\hat{x}\hat{p}_y - \hat{y}\hat{p}_x, \hat{p}_y] = [\hat{x}\hat{p}_y, \hat{p}_y] - [\hat{y}\hat{p}_x, \hat{p}_y] = 0 - [\hat{y}, \hat{p}_y]\hat{p}_x = -i\hbar\hat{p}_x, \\
[\hat{L}_z, \hat{p}_z] &= [\hat{x}\hat{p}_y - \hat{y}\hat{p}_x, \hat{p}_z] = [\hat{x}\hat{p}_y, \hat{p}_z] - [\hat{y}\hat{p}_x, \hat{p}_z] = 0 - 0 = 0.
\end{aligned}
$$

(3)

$$
\begin{aligned}
[\hat{L}_z, \hat{r}^2] &= [\hat{L}_z, \hat{x}^2 + \hat{y}^2 + \hat{z}^2] \\
&= \hat{x}[\hat{L}_z, \hat{x}] + [\hat{L}_z, \hat{x}]\hat{x} + \hat{y}[\hat{L}_z, \hat{y}] + [\hat{L}_z, \hat{y}]\hat{y} + \hat{z}[\hat{L}_z, \hat{z}] + [\hat{L}_z, \hat{z}]\hat{z} \\
&= i\hbar\hat{x}\hat{y} + i\hbar\hat{y}\hat{x} - i\hbar\hat{y}\hat{x} - i\hbar\hat{x}\hat{y} + 0 + 0 = 0, \\
[\hat{L}_z, \hat{p}^2] &= [\hat{L}_z, \hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2] \\
&= \hat{p}_x[\hat{L}_z, \hat{p}_x] + [\hat{L}_z, \hat{p}_x]\hat{p}_x + \hat{p}_y[\hat{L}_z, \hat{p}_y] + [\hat{L}_z, \hat{p}_y]\hat{p}_y + \hat{p}_z[\hat{L}_z, \hat{p}_z] + [\hat{L}_z, \hat{p}_z]\hat{p}_z \\
&= i\hbar\hat{p}_x\hat{p}_y + i\hbar\hat{p}_y\hat{p}_x - i\hbar\hat{p}_y\hat{p}_x - i\hbar\hat{p}_x\hat{p}_y + 0 + 0 = 0.
\end{aligned}
$$

[問 2]　昇降演算子 $\hat{L}_\pm := \hat{L}_x \pm i\hat{L}_y$ について、

$$
\hat{L}_+|l,m\rangle = A_l^m|l,m+1\rangle, \qquad \hat{L}_-|l,m\rangle = B_l^m|l,m-1\rangle.
$$

に現れる定数 A_l^m, B_l^m を以下の手続きで求めよ。ただし状態 $|l,m\rangle$, $|l,m\pm 1\rangle$ は規格化されているものとし、A_l^m, B_l^m は正の実数とする。

(1)　昇降演算子が

$$
\hat{L}_\pm^\dagger = \hat{L}_\mp,
$$

を満たすことを示せ。

(2)　昇降演算子の積が

$$
\hat{L}_\mp\hat{L}_\pm = \hat{L}^2 - \hat{L}_z^2 \mp \hbar\hat{L}_z,
$$

を満たすことを示せ。ただし $\hat{L}^2 := \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$ である。

(3)　(2) の結果に左右から $\langle l,m|\cdots|l,m\rangle$ を作用させることで、

$$
\begin{aligned}
A_l^m &= \hbar\sqrt{l(l+1) - m(m+1)} \ \left(= \hbar\sqrt{(l-m)(l+m+1)}\right), \\
B_l^m &= \hbar\sqrt{l(l+1) - m(m-1)} \ \left(= \hbar\sqrt{(l+m)(l-m+1)}\right),
\end{aligned}
$$

を示せ。

[解 2]

(1) $\hat{L}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y$ および $\hat{L}_y = \hat{z}\hat{p}_x - \hat{x}\hat{p}_z$ より

$$\hat{L}_x^\dagger = \hat{p}_z^\dagger \hat{y}^\dagger - \hat{p}_y^\dagger \hat{z}^\dagger \overset{\hat{y},\ \hat{z},\ \hat{p}_y,\ \hat{p}_z\text{の Hermite 性}}{=} \hat{p}_z\hat{y} - \hat{p}_y\hat{z} \overset{[\hat{y},\hat{p}_z]=0,\ [\hat{z},\hat{p}_y]=0}{=} \hat{y}\hat{p}_z - \hat{z}\hat{p}_y = \hat{L}_x,$$
$$\hat{L}_y^\dagger = \hat{p}_x^\dagger \hat{z}^\dagger - \hat{p}_z^\dagger \hat{x}^\dagger \overset{\hat{z},\ \hat{x},\ \hat{p}_z,\ \hat{p}_x\text{の Hermite 性}}{=} \hat{p}_x\hat{z} - \hat{p}_z\hat{x} \overset{[\hat{z},\hat{p}_x]=0,\ [\hat{x},\hat{p}_z]=0}{=} \hat{z}\hat{p}_x - \hat{x}\hat{p}_z = \hat{L}_y,$$

より $\hat{L}_x,\ \hat{L}_y$ は Hermite である。よって

$$\hat{L}_\pm^\dagger = \left(\hat{L}_x \pm i\hat{L}_y\right)^\dagger = \hat{L}_x \mp i\hat{L}_y = \hat{L}_\mp,$$

となる。

(2) 昇降演算子の定義および交換関係から

$$\hat{L}_\mp\hat{L}_\pm = \left(\hat{L}_x \mp i\hat{L}_y\right)\left(\hat{L}_x \pm i\hat{L}_y\right) = \hat{L}_x^2 + \hat{L}_y^2 \pm i[\hat{L}_x, \hat{L}_y] \overset{[\hat{L}_x,\hat{L}_y]=i\hbar\hat{L}_z}{=} \hat{L}_x^2 + \hat{L}_y^2 \mp \hbar\hat{L}_z \overset{\hat{L}^2:=\hat{L}_x^2+\hat{L}_y^2+\hat{L}_z^2}{=} \hat{L}^2 - \hat{L}_z^2 \mp \hbar\hat{L}_z.$$

となる。

(3) (1) より

$$\langle l, m|\hat{L}_\mp\hat{L}_\pm|l, m\rangle = \left|\hat{L}_\pm|l,\ m\rangle\right|^2 = (A_l^m)^2 \text{ or } (B_l^m)^2,$$

である一方、(2) より

$$\langle l, m|\hat{L}_\mp\hat{L}_\pm|l, m\rangle = \langle l, m|\left(\hat{L}^2 - \hat{L}_z^2 \mp \hbar\hat{L}_z\right)|l, m\rangle = \hbar^2\left[l(l+1) - m^2 \mp m\right] = \hbar^2\left[l(l+1) - m(m\pm1)\right],$$

であるから、

$$A_l^m = \hbar\sqrt{l(l+1) - m(m+1)}\ \left(= \hbar\sqrt{(l-m)(l+m+1)}\right),$$
$$B_l^m = \hbar\sqrt{l(l+1) - m(m-1)}\ \left(= \hbar\sqrt{(l+m)(l-m+1)}\right),$$

となる。

[問 3] 全角運動量の二乗に対する表式

$$L^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right],$$

を導こう。

(1) 角運動量

$$\boldsymbol{L} = -i\hbar\,\boldsymbol{r}\times\nabla,$$

に対し、$\boldsymbol{r} = r\boldsymbol{e}_r$ および

$$\nabla = \boldsymbol{e}_r\frac{\partial}{\partial r} + \boldsymbol{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \boldsymbol{e}_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi},$$

を用いて、

$$\boldsymbol{L} = -i\hbar\left(\boldsymbol{e}_\phi \frac{\partial}{\partial\theta} - \boldsymbol{e}_\theta \frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right),$$

を示せ。

(2) (1) の結果に、極座標における基底ベクトルと直交座標における基底ベクトルの関係

$$\begin{aligned}
\boldsymbol{e}_r &= \sin\theta\cos\phi\,\boldsymbol{e}_x + \sin\theta\sin\phi\,\boldsymbol{e}_y + \cos\theta\,\boldsymbol{e}_z,\\
\boldsymbol{e}_\theta &= \cos\theta\cos\phi\,\boldsymbol{e}_x + \cos\theta\sin\phi\,\boldsymbol{e}_y - \sin\theta\,\boldsymbol{e}_z,\\
\boldsymbol{e}_\phi &= -\sin\phi\,\boldsymbol{e}_x + \cos\phi\,\boldsymbol{e}_y,
\end{aligned}$$

を用いると

$$\boldsymbol{L} = -i\hbar\left[\boldsymbol{e}_x\left(-\sin\phi\frac{\partial}{\partial\theta} - \frac{\cos\phi}{\tan\theta}\frac{\partial}{\partial\phi}\right) + \boldsymbol{e}_y\left(\cos\phi\frac{\partial}{\partial\theta} - \frac{\sin\phi}{\tan\theta}\frac{\partial}{\partial\phi}\right) + \boldsymbol{e}_z\frac{\partial}{\partial\phi}\right],$$

となる。これを用いて、昇降演算子の表式

$$L_\pm = L_x \pm iL_y = \pm\hbar e^{\pm i\phi}\left(\frac{\partial}{\partial\theta} \pm \frac{i}{\tan\theta}\frac{\partial}{\partial\phi}\right),$$

を示せ。

(3) 昇降演算子の積が

$$L_\pm L_\mp = -\hbar^2\left(\frac{\partial^2}{\partial\theta^2} + \frac{1}{\tan\theta}\frac{\partial}{\partial\theta} + \frac{1}{\tan^2\theta}\frac{\partial^2}{\partial\phi^2} \pm i\frac{\partial}{\partial\phi}\right),$$

となることを示し、$L^2 = L_\pm L_\mp + L_z^2 \mp \hbar L_z$ と合わせて

$$L^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right],$$

を示せ。

[解 3]

(1) 題意より

$$\begin{aligned}
\boldsymbol{L} &= -i\hbar r\boldsymbol{e}_r \times \left(\boldsymbol{e}_r\frac{\partial}{\partial r} + \boldsymbol{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \boldsymbol{e}_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\\
&= -i\hbar r\left((\boldsymbol{e}_r \times \boldsymbol{e}_r)\frac{\partial}{\partial r} + (\boldsymbol{e}_r \times \boldsymbol{e}_\theta)\frac{1}{r}\frac{\partial}{\partial\theta} + (\boldsymbol{e}_r \times \boldsymbol{e}_\phi)\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\\
&= -i\hbar r\left(0\frac{\partial}{\partial r} + \boldsymbol{e}_\phi\frac{1}{r}\frac{\partial}{\partial\theta} - \boldsymbol{e}_\theta\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\\
&= -i\hbar\left(\boldsymbol{e}_\phi\frac{\partial}{\partial\theta} - \boldsymbol{e}_\theta\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right),
\end{aligned}$$

となる。

(2) 題意より

$$L_x = -i\hbar\left(-\sin\phi\frac{\partial}{\partial\theta} - \frac{\cos\phi}{\tan\theta}\frac{\partial}{\partial\phi}\right), \qquad L_y = -i\hbar\left(\cos\phi\frac{\partial}{\partial\theta} - \frac{\sin\phi}{\tan\theta}\frac{\partial}{\partial\phi}\right), \qquad L_z = -i\hbar\frac{\partial}{\partial\phi},$$

であるから、昇降演算子は

$$L_\pm = L_x \pm iL_y = -i\hbar\left(-\sin\phi\frac{\partial}{\partial\theta} - \frac{\cos\phi}{\tan\theta}\frac{\partial}{\partial\phi}\right) \pm \hbar\left(\cos\phi\frac{\partial}{\partial\theta} - \frac{\sin\phi}{\tan\theta}\frac{\partial}{\partial\phi}\right) = \pm\hbar e^{\pm i\phi}\left(\frac{\partial}{\partial\theta} \pm \frac{i}{\tan\theta}\frac{\partial}{\partial\phi}\right),$$

となる。

(3)　(3) の結果より、昇降演算子の積は

$$L_\pm L_\mp = -\hbar^2\left(\frac{\partial^2}{\partial\theta^2} + \frac{1}{\tan\theta}\frac{\partial}{\partial\theta} + \frac{1}{\tan^2\theta}\frac{\partial^2}{\partial\phi^2} \pm i\frac{\partial}{\partial\phi}\right),$$

となる。これと $L^2 = L_\pm L_\mp + L_z^2 \mp \hbar L_z$ より、

$$\begin{aligned}
L^2 &= -\hbar^2\left(\frac{\partial^2}{\partial\theta^2} + \frac{1}{\tan\theta}\frac{\partial}{\partial\theta} + \frac{1}{\tan^2\theta}\frac{\partial^2}{\partial\phi^2} \pm i\frac{\partial}{\partial\phi}\right) + \left(-i\hbar\frac{\partial}{\partial\phi}\right)^2 \mp \hbar\left(-i\hbar\frac{\partial}{\partial\phi}\right) \\
&= -\hbar^2\left(\frac{\partial^2}{\partial\theta^2} + \frac{1}{\tan\theta}\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right) \\
&= -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right],
\end{aligned}$$

を得る。