

演習問題

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学籍番号

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[問 1] 角運動量演算子について、以下の問に答えよ。

(1) $[\hat{x}, \hat{p}_x] = i\hbar$, $[\hat{y}, \hat{p}_y] = i\hbar$, $[\hat{z}, \hat{p}_z] = i\hbar$ を用いて、交換関係

$$[\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z, \quad [\hat{L}_y, \hat{L}_z] = i\hbar\hat{L}_x, \quad [\hat{L}_z, \hat{L}_x] = i\hbar\hat{L}_y,$$

を示せ。 $[\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z$ についてのみ示せばよい。

(2) 交換関係

$$\begin{aligned} [\hat{L}_z, \hat{x}] &= i\hbar\hat{y}, & [\hat{L}_z, \hat{y}] &= -i\hbar\hat{x}, & [\hat{L}_z, \hat{z}] &= 0, \\ [\hat{L}_z, \hat{p}_x] &= i\hbar\hat{p}_y, & [\hat{L}_z, \hat{p}_y] &= -i\hbar\hat{p}_x, & [\hat{L}_z, \hat{p}_z] &= 0, \end{aligned}$$

を示せ。

(3) 交換関係

$$[\hat{L}_z, \hat{r}^2] = 0, \quad [\hat{L}_z, \hat{p}^2] = 0,$$

を示せ。ただし $\hat{r}^2 := \hat{x}^2 + \hat{y}^2 + \hat{z}^2$, $\hat{p}^2 := \hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2$ である。

[解 1]

(1)

$$[\hat{L}_x, \hat{L}_y] = [\hat{y}\hat{p}_z - \hat{z}\hat{p}_y, \hat{z}\hat{p}_x - \hat{x}\hat{p}_z] = [\hat{y}\hat{p}_z, \hat{z}\hat{p}_x] - [\hat{y}\hat{p}_z, \hat{x}\hat{p}_z] - [\hat{z}\hat{p}_y, \hat{z}\hat{p}_x] + [\hat{z}\hat{p}_y, \hat{x}\hat{p}_z],$$

であるが、右辺のそれぞれの項は

$$[\hat{y}\hat{p}_z, \hat{z}\hat{p}_x] = \hat{z}[\hat{y}\hat{p}_z, \hat{p}_x] + [\hat{y}\hat{p}_z, \hat{z}]\hat{p}_x = 0 + [\hat{y}\hat{p}_z, \hat{z}]\hat{p}_x = (\hat{y}[\hat{p}_z, \hat{z}] + [\hat{y}, \hat{z}]\hat{p}_z)\hat{p}_x = -i\hbar\hat{y}\hat{p}_x,$$

$$[\hat{y}\hat{p}_z, \hat{x}\hat{p}_z] = \hat{x}[\hat{y}\hat{p}_z, \hat{p}_z] + [\hat{y}\hat{p}_z, \hat{x}]\hat{p}_z = 0 + 0 = 0,$$

$$[\hat{z}\hat{p}_y, \hat{z}\hat{p}_x] = \hat{z}[\hat{z}\hat{p}_y, \hat{p}_x] + [\hat{z}\hat{p}_y, \hat{z}]\hat{p}_x = 0 + 0 = 0,$$

$$[\hat{z}\hat{p}_y, \hat{x}\hat{p}_z] = \hat{x}[\hat{z}\hat{p}_y, \hat{p}_z] + [\hat{z}\hat{p}_y, \hat{x}]\hat{p}_z = \hat{x}[\hat{z}\hat{p}_y, \hat{p}_z] + 0 = \hat{x}(\hat{z}[\hat{p}_y, \hat{p}_z] + [\hat{z}, \hat{p}_z]\hat{p}_y) = i\hbar\hat{x}\hat{p}_y,$$

であるから、

$$[\hat{L}_x, \hat{L}_y] = i\hbar(\hat{x}\hat{p}_y - \hat{y}\hat{p}_x) = i\hbar\hat{L}_z,$$

となる。他も同様である。

(2)

$$\begin{aligned}
[\hat{L}_z, \hat{x}] &= [\hat{x}\hat{p}_y - \hat{y}\hat{p}_x, \hat{x}] = [\hat{x}\hat{p}_y, \hat{x}] - [\hat{y}\hat{p}_x, \hat{x}] = 0 - \hat{y}[\hat{p}_x, \hat{x}] = i\hbar\hat{y}, \\
[\hat{L}_z, \hat{y}] &= [\hat{x}\hat{p}_y - \hat{y}\hat{p}_x, \hat{y}] = [\hat{x}\hat{p}_y, \hat{y}] - [\hat{y}\hat{p}_x, \hat{y}] = \hat{x}[\hat{p}_y, \hat{y}] - 0 = -i\hbar\hat{x}, \\
[\hat{L}_z, \hat{z}] &= [\hat{x}\hat{p}_y - \hat{y}\hat{p}_x, \hat{z}] = [\hat{x}\hat{p}_y, \hat{z}] - [\hat{y}\hat{p}_x, \hat{z}] = 0 - 0 = 0, \\
[\hat{L}_z, \hat{p}_x] &= [\hat{x}\hat{p}_y - \hat{y}\hat{p}_x, \hat{p}_x] = [\hat{x}\hat{p}_y, \hat{p}_x] - [\hat{y}\hat{p}_x, \hat{p}_x] = [\hat{x}, \hat{p}_x]\hat{p}_y - 0 = i\hbar\hat{p}_y, \\
[\hat{L}_z, \hat{p}_y] &= [\hat{x}\hat{p}_y - \hat{y}\hat{p}_x, \hat{p}_y] = [\hat{x}\hat{p}_y, \hat{p}_y] - [\hat{y}\hat{p}_x, \hat{p}_y] = 0 - [\hat{y}, \hat{p}_y]\hat{p}_x = -i\hbar\hat{p}_x, \\
[\hat{L}_z, \hat{p}_z] &= [\hat{x}\hat{p}_y - \hat{y}\hat{p}_x, \hat{p}_z] = [\hat{x}\hat{p}_y, \hat{p}_z] - [\hat{y}\hat{p}_x, \hat{p}_z] = 0 - 0 = 0.
\end{aligned}$$

(3)

$$\begin{aligned}
[\hat{L}_z, \hat{r}^2] &= [\hat{L}_z, \hat{x}^2 + \hat{y}^2 + \hat{z}^2] \\
&= \hat{x}[\hat{L}_z, \hat{x}] + [\hat{L}_z, \hat{x}]\hat{x} + \hat{y}[\hat{L}_z, \hat{y}] + [\hat{L}_z, \hat{y}]\hat{y} + \hat{z}[\hat{L}_z, \hat{z}] + [\hat{L}_z, \hat{z}]\hat{z} \\
&= i\hbar\hat{x}\hat{y} + i\hbar\hat{y}\hat{x} - i\hbar\hat{y}\hat{x} - i\hbar\hat{x}\hat{y} + 0 + 0 = 0, \\
[\hat{L}_z, \hat{p}^2] &= [\hat{L}_z, \hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2] \\
&= \hat{p}_x[\hat{L}_z, \hat{p}_x] + [\hat{L}_z, \hat{p}_x]\hat{p}_x + \hat{p}_y[\hat{L}_z, \hat{p}_y] + [\hat{L}_z, \hat{p}_y]\hat{p}_y + \hat{p}_z[\hat{L}_z, \hat{p}_z] + [\hat{L}_z, \hat{p}_z]\hat{p}_z \\
&= i\hbar\hat{p}_x\hat{p}_y + i\hbar\hat{p}_y\hat{p}_x - i\hbar\hat{p}_y\hat{p}_x - i\hbar\hat{p}_x\hat{p}_y + 0 + 0 = 0.
\end{aligned}$$

[問 2] 昇降演算子 $\hat{L}_\pm := \hat{L}_x \pm i\hat{L}_y$ について、

$$\hat{L}_+ |l, m\rangle = A_l^m |l, m+1\rangle, \quad \hat{L}_- |l, m\rangle = B_l^m |l, m-1\rangle.$$

に現れる定数 A_l^m, B_l^m を以下の手続きで求めよ。ただし状態 $|l, m\rangle, |l, m \pm 1\rangle$ は規格化されているものとし、 A_l^m, B_l^m は正の実数とする。

(1) 昇降演算子が

$$\hat{L}_\pm^\dagger = \hat{L}_\mp,$$

を満たすことを示せ。

(2) 昇降演算子の積が

$$\hat{L}_\mp \hat{L}_\pm = \hat{L}^2 - \hat{L}_z^2 \mp \hbar \hat{L}_z,$$

を満たすことを示せ。ただし $\hat{L}^2 := \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$ である。

(3) (2) の結果に左右から $\langle l, m | \cdots | l, m \rangle$ を作用させることで、

$$\begin{aligned}
A_l^m &= \hbar \sqrt{l(l+1) - m(m+1)} \quad (= \hbar \sqrt{(l-m)(l+m+1)}), \\
B_l^m &= \hbar \sqrt{l(l+1) - m(m-1)} \quad (= \hbar \sqrt{(l+m)(l-m+1)}),
\end{aligned}$$

を示せ。

[解 2]

(1) $\hat{L}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y$ および $\hat{L}_y = \hat{z}\hat{p}_x - \hat{x}\hat{p}_z$ より

$$\begin{aligned}\hat{L}_x^\dagger &= \hat{p}_z^\dagger \hat{y}^\dagger - \hat{p}_y^\dagger \hat{z}^\dagger \stackrel{\text{Hermite 性}}{=} \hat{p}_z \hat{y} - \hat{p}_y \hat{z} \stackrel{[\hat{y}, \hat{p}_z]=0, [\hat{z}, \hat{p}_y]=0}{=} \hat{y}\hat{p}_z - \hat{z}\hat{p}_y = \hat{L}_x, \\ \hat{L}_y^\dagger &= \hat{p}_x^\dagger \hat{z}^\dagger - \hat{p}_z^\dagger \hat{x}^\dagger \stackrel{\text{Hermite 性}}{=} \hat{p}_x \hat{z} - \hat{p}_z \hat{x} \stackrel{[\hat{z}, \hat{p}_x]=0, [\hat{x}, \hat{p}_z]=0}{=} \hat{z}\hat{p}_x - \hat{x}\hat{p}_z = \hat{L}_y,\end{aligned}$$

より \hat{L}_x, \hat{L}_y は Hermite である。よって

$$\hat{L}_\pm^\dagger = (\hat{L}_x \pm i\hat{L}_y)^\dagger = \hat{L}_x \mp i\hat{L}_y = \hat{L}_\mp,$$

となる。

(2) 昇降演算子の定義および交換関係から

$$\hat{L}_\mp \hat{L}_\pm = (\hat{L}_x \mp i\hat{L}_y)(\hat{L}_x \pm i\hat{L}_y) = \hat{L}_x^2 + \hat{L}_y^2 \pm i[\hat{L}_x, \hat{L}_y] \stackrel{[\hat{L}_x, \hat{L}_y]=i\hbar\hat{L}_z}{=} \hat{L}_x^2 + \hat{L}_y^2 \mp \hbar\hat{L}_z \stackrel{\hat{L}^2 := \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2}{=} \hat{L}^2 - \hat{L}_z^2 \mp \hbar\hat{L}_z.$$

となる。

(3) (1) より

$$\langle l, m | \hat{L}_\mp \hat{L}_\pm | l, m \rangle = |\hat{L}_\pm | l, m \rangle|^2 = (A_l^m)^2 \text{ or } (B_l^m)^2,$$

である一方、(2) より

$$\langle l, m | \hat{L}_\mp \hat{L}_\pm | l, m \rangle = \langle l, m | (\hat{L}^2 - \hat{L}_z^2 \mp \hbar\hat{L}_z) | l, m \rangle = \hbar^2 [l(l+1) - m^2 \mp m] = \hbar^2 [l(l+1) - m(m \pm 1)],$$

であるから、

$$\begin{aligned}A_l^m &= \hbar\sqrt{l(l+1) - m(m+1)} \quad (= \hbar\sqrt{(l-m)(l+m+1)}), \\ B_l^m &= \hbar\sqrt{l(l+1) - m(m-1)} \quad (= \hbar\sqrt{(l+m)(l-m+1)}),\end{aligned}$$

となる。

[問 3] 全角運動量の二乗に対する表式

$$L^2 = -\hbar^2 \left[\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right],$$

を導こう。

(1) 角運動量

$$\mathbf{L} = -i\hbar \mathbf{r} \times \nabla,$$

に対し、 $\mathbf{r} = r\mathbf{e}_r$ および

$$\nabla = \mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r} \frac{\partial}{\partial\theta} + \mathbf{e}_\phi \frac{1}{r\sin\theta} \frac{\partial}{\partial\phi},$$

を用いて、

$$L = -i\hbar \left(e_\phi \frac{\partial}{\partial \theta} - e_\theta \frac{1}{\sin \theta} \frac{\partial}{\partial \phi} \right),$$

を示せ。

(2) (1)の結果に、球座標における基底ベクトルと直交座標における基底ベクトルの関係

$$e_r = \sin \theta \cos \phi e_x + \sin \theta \sin \phi e_y + \cos \theta e_z,$$

$$e_\theta = \cos \theta \cos \phi e_x + \cos \theta \sin \phi e_y - \sin \theta e_z,$$

$$e_\phi = -\sin \phi e_x + \cos \phi e_y,$$

を用いると

$$L = -i\hbar \left[e_x \left(-\sin \phi \frac{\partial}{\partial \theta} - \frac{\cos \phi}{\tan \theta} \frac{\partial}{\partial \phi} \right) + e_y \left(\cos \phi \frac{\partial}{\partial \theta} - \frac{\sin \phi}{\tan \theta} \frac{\partial}{\partial \phi} \right) + e_z \frac{\partial}{\partial \phi} \right],$$

となる。これを用いて、昇降演算子の表式

$$L_\pm = L_x \pm iL_y = \pm \hbar e^{\pm i\phi} \left(\frac{\partial}{\partial \theta} \pm \frac{i}{\tan \theta} \frac{\partial}{\partial \phi} \right),$$

を示せ。

(3) 昇降演算子の積が

$$L_\pm L_\mp = -\hbar^2 \left(\frac{\partial^2}{\partial \theta^2} + \frac{1}{\tan \theta} \frac{\partial}{\partial \theta} + \frac{1}{\tan^2 \theta} \frac{\partial^2}{\partial \phi^2} \pm i \frac{\partial}{\partial \phi} \right),$$

となることを示し、 $L^2 = L_\pm L_\mp + L_z^2 \mp \hbar L_z$ と合わせて

$$L^2 = -\hbar^2 \left[\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right],$$

を示せ。

[解 3]

(1) 題意より

$$\begin{aligned} L &= -i\hbar r e_r \times \left(e_r \frac{\partial}{\partial r} + e_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + e_\phi \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \right) \\ &= -i\hbar r \left((e_r \times e_r) \frac{\partial}{\partial r} + (e_r \times e_\theta) \frac{1}{r} \frac{\partial}{\partial \theta} + (e_r \times e_\phi) \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \right) \\ &= -i\hbar r \left(0 \frac{\partial}{\partial r} + e_\phi \frac{1}{r} \frac{\partial}{\partial \theta} - e_\theta \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \right) \\ &= -i\hbar \left(e_\phi \frac{\partial}{\partial \theta} - e_\theta \frac{1}{\sin \theta} \frac{\partial}{\partial \phi} \right), \end{aligned}$$

となる。

(2) 題意より

$$L_x = -i\hbar \left(-\sin \phi \frac{\partial}{\partial \theta} - \frac{\cos \phi}{\tan \theta} \frac{\partial}{\partial \phi} \right), \quad L_y = -i\hbar \left(\cos \phi \frac{\partial}{\partial \theta} - \frac{\sin \phi}{\tan \theta} \frac{\partial}{\partial \phi} \right), \quad L_z = -i\hbar \frac{\partial}{\partial \phi},$$

であるから、昇降演算子は

$$L_{\pm} = L_x \pm iL_y = -i\hbar \left(-\sin\phi \frac{\partial}{\partial\theta} - \frac{\cos\phi}{\tan\theta} \frac{\partial}{\partial\phi} \right) \pm \hbar \left(\cos\phi \frac{\partial}{\partial\theta} - \frac{\sin\phi}{\tan\theta} \frac{\partial}{\partial\phi} \right) = \pm\hbar e^{\pm i\phi} \left(\frac{\partial}{\partial\theta} \pm \frac{i}{\tan\theta} \frac{\partial}{\partial\phi} \right),$$

となる。

(3) (3)の結果より、昇降演算子の積は

$$L_{\pm}L_{\mp} = -\hbar^2 \left(\frac{\partial^2}{\partial\theta^2} + \frac{1}{\tan\theta} \frac{\partial}{\partial\theta} + \frac{1}{\tan^2\theta} \frac{\partial^2}{\partial\phi^2} \pm i \frac{\partial}{\partial\phi} \right),$$

となる。これと $L^2 = L_{\pm}L_{\mp} + L_z^2 \mp \hbar L_z$ より、

$$\begin{aligned} L^2 &= -\hbar^2 \left(\frac{\partial^2}{\partial\theta^2} + \frac{1}{\tan\theta} \frac{\partial}{\partial\theta} + \frac{1}{\tan^2\theta} \frac{\partial^2}{\partial\phi^2} \pm i \frac{\partial}{\partial\phi} \right) + \left(-i\hbar \frac{\partial}{\partial\phi} \right)^2 \mp \hbar \left(-i\hbar \frac{\partial}{\partial\phi} \right) \\ &= -\hbar^2 \left(\frac{\partial^2}{\partial\theta^2} + \frac{1}{\tan\theta} \frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right) \\ &= -\hbar^2 \left[\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right], \end{aligned}$$

を得る。